FROM LEIBNIZ'S *CHARACTERISTICA GEOMETRICA* TO CONTEMPORARY GEOMETRIC ALGEBRA

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1.- Foreword.

The view advanced in this paper is that *Geometric algebra* (GA) is a mathematical structure that allows the representation of geometrical concepts, and operations on them, in an intrinsic manner (it is a coordinate-free system), and that its syntax and semantics are logically rigorous and yet not inimical to intuition nor to sophisticated applications. In our view, these features are in tune with Leibniz's ideas on the subject, even though, as we shall argue, their scope was necessarily relatively limited at his time.

Many researchers have contributed to the rather slow unfolding of GA, often in scientific and engineering applications in which geometry plays a key role. Here is a sample of a few landmark texts produced in the last half a century: Chevalley (1956), Riesz (1958), Hestenes (1966, 2015), Hestenes (1990, 1999), Hestenes-Sobczyk (1984), Lounesto (1997), Doran-Lasenby (2003), Dorst-Fontijne-Mann (2007), Snygg (2012), Rodrigues-Oliveira (2016).

However, any newcomer to this rich field will learn, sooner or later, that GA has a much longer history. Fundamental and systematic works were published since the beginning of the XIX century, but, most importantly for our purposes here, it was Leibniz who put forward the seminal ideas, in what he called *characteristica geometrica*[1], as early as the beginning of the last quarter of the XVII century. This was well one century and a half before the issue resurfaced again. It is natural, therefore, to ask about the *impact* of Leibniz ideas in those later developments. It is also natural to consider *to what degree have they been fulfilled today*.

1 In today's terms, *characteristica* amounts to *symbolism*, or *symbolic calculus*.

The main purpose of this paper is to try to shed some light on both issues. This is not as straightforward as it might seem. Regarding the first question, we can document only one key influence, which happens to be only an indirect one (Grassmann 1847), but we cannot refrain from conjecturing that there must have been others, most likely caused by the powerful irradiation of Leibniz's thinking, and writing, on all sorts of topics. For Burali-Forti 1897b, for example, "Leibniz's idea was destined to propagate and to produce great results", and mentions names that did important work before Grassmann, such as Caspar Vessel and Giusto Bellavitis. One likely propagation process may have been through philosophy and theology. Grassmann was a theologian and a linguist, with no university education in mathematics, and to a good extent, he received his ideas about mathematical innovation from Friedrich D. E. Schleiermacher (1768-1834), who was a philosopher, classical philologist and theologian (see Achtner 2016). Fundamentally, it came about through the study of Schleiermacher's teachings, among which it is worth singling out not only his extensive writings on dialectics, but also his 1831 lecture on Leibniz's idea of a universal language (Forster 2015). Those teachings "were picked up by Grassmann and operationalized in his philosophical-mathematical treatise *Ausdehnungslehre* in 1844" (Achtner 2016).

The main obstacle to assess the second question is that there is not a clear consensus on what the geometric algebra structure is. The main source of tension is the unfortunate divide between abstract and general mathematical presentations, that tend to eschew any sort of applications, and those by authors whose main motivation lies in physics or engineering, that tend to be regarded by the other side as lacking rigor. Our approach will be to outline a structure that retains the positive aspects of both sides. Our intention is none other than to facilitate, whatever the background of the reader, to hook on it to understand the other backgrounds. Thus, a mathematician can get a good understanding of physical theories based on her mathematical knowledge and a physicist can get a better appreciation of the mathematics by standing on his understanding of the physics. One may even indulge in the fancy that Leibniz himself would have judged the system as a flexible embodiment of his *characteristica geometrica*.

2.- On the characteristica universalis.

In Russell 1946, chapter on Leibniz, we find (pages 572-573) an authoritative and general view about the pioneering ideas of Leibniz on logic and mathematics:

> *"Leibniz was a firm believer in the importance of logic, not only in its own sphere, but* [also] *as a basis of metaphysics. He did work on mathematical logic, which would have been enormously important if he had published it; he would, in that case, have been the founder of mathematical logic, which would have become known a century and a half sooner than it did in fact. He abstained from publishing, because he kept on finding evidence that Aristotle's doctrine of the syllogism was wrong on some points; respect for Aristotle made it impossible for him to believe this, so he mistakenly supposed that the errors must be his own.*
>
> *Nevertheless he cherished throughout his life the hope of discovering a kind of generalized mathematics, which he called* characteristica universalis *(CU), by means of which thinking could be replaced by calculation. 'If we had it,' he says, 'we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis.' 'If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other (with a friend as witness, if they liked): Let us calculate (*Calculemus*)".*

The closest idea in today's world that has an appearance of a CU is what may be loosely called *pattern theory.* Although it appears in many guises in all sorts of developments occurred in the last decades (cf. Grenander 1996, 2012), my interest here is the general views expressed by David Mumford and which I regard as representative of those held by most workers in that discipline. In Mumford 1992 we find the following view (the **boldface** and **underlined**, are not in the original):

> *"In summary, my belief is that pattern theory contains the germs of* ***a universal theory of thought itself, one that stands in opposition to the accepted analysis of thought in terms of logic.*** <u>*The successes to date of the theory are certainly insufficient to justify such a grandiose dream, but no other theory has been more successful.*</u> *The*

> *extraordinary similarity of the structure of all parts of the human cortex to each other and of human cortex with the cortex of the most primitive mammals suggests that* ***a relatively simple universal principal governs its operation****, even in complex processes like language:* ***pattern theory is a proposal for what these principles may be****"*.

The paper Mumford 1996 is an edited version of Mumford 1992, and the italic text in the quotation was deleted. In my biographical paper Xambó-Descamps 2013, I asked Mumford whether he still held, two decades after that paper, that pattern theory "contains the germs of a universal theory of thought" and how did this relate to "other universal languages that have recurrently been proposed in the past". His answer was this: "At the time of that lecture, I didn't appreciate fully the importance of graphical structures underlying thought. I feel the identification of feedback in cortex with the prior in Bayes' formula is quite correct but I have since pursued the relevance of grammar like graphs, e.g. in my monograph, with S-C Zhu, *A stochastic grammar of images: Foundations and Trends in Computer Graphics and Vision* (2007, 259-362). But I continue to argue strongly that logic based models are inadequate". Asked about the progress in the understanding of the neocortex, he said that the basic cortical "algorithms" remain a mystery and that the difficulty of recording the operation of something like a million neurons is a great practical obstacle.

This, and other developments related to linguistics and artificial intelligence, lead me to regard Leibniz's phrasings of his *characteristica universalis* as fundamentally utopic, which does not mean that they cannot be useful as an unreachable horizon in the sense explained in the abstract of this paper, included at the end of this volume.

3.- On the *characteristica geometrica*.

Leibniz's idea of a *characteristica geometrica* (CG), a (mathematical) branch of the CU, was advanced to Christian Huygens in a letter, sent together with an essay (both in French), dated September 8, 1679. The first paragraph of next quotation is from the letter and the second and third from the essay (Leibniz 1850, 1956; cf. Crowe 1994, 3-4, that quotes the English translation):

> *"But after all the progress I have made in these matters* [in going beyond Viète and Descartes], *I am not yet satisfied with Algebra, because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, as far as geometry is concerned, we need still another analysis, which is distinctly geometrical or linear and which will express* situation *directly as algebra expresses* magnitude *directly. I believe that I have found the way, and that we can represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes. I believe that by this method one could treat mechanics almost like geometry. I am sending you an essay, which seems to me to be important. […]*
>
> *I have discovered certain elements of a new characteristic which is entirely different from algebra and which will have great advantages in representing to the mind, exactly and in a way faithful to its nature, even without figures, everything which depends on sense perception. Algebra is the characteristic for undetermined numbers or magnitudes only, but it does not express situation, angles, and motion directly. Hence it is often difficult to analyze the properties of a figure by calculation, and still more difficult to find very convenient geometrical demonstrations and constructions, even when the algebraic calculation is completed. But this new characteristic, which follows the visual figures, cannot fail to give the solution, the construction, and the geometric demonstration all at the same time, and in a natural way in one analysis, that is, through determined procedure.*
>
> *But its chief value lies in the reasoning which can be done and the conclusions that can be drawn by operations with its characters, which could not be expressed in figures, and still less in models, without multiplying these too greatly or without confusing them with too many points and lines in the course of the many futile attempts one is forced to make. This method, by contrast, will guide us surely and without effort. I believe that by this method one could treat mechanics almost like geometry, and one could even test the qualities of materials, because this ordinarily depends on certain figures in their sensible parts. Finally, I have no hope that we can get very far in physics until we have found some such method of abridgement to lighten its burden of imagination"*.

All those claims must have appeared as too ambitious or even groundless, to contemporaries, and in particular to Huygens. In retrospect, however, they are natural thoughts of a mind in possession of a powerful model of logic, construed (see Antognazza 2009, 233 ff.) as tools of valid reasoning, to be used by all other sciences (*scientia generalis*), to discover (*ars inveniendi*) and to demonstrate from sufficient data (*ars judicandi*), all driven by a universal symbolism (*characteristica universlis*). It seems clear that his transforming vision in the case of mathematics was not limited to what is usually recognized (Euclidean geometry, Arithmetic, Vieta's symbolic algebra/analysis (1591), Descartes' analytical geometry (1637)), but that it included the basic ideas on projective geometry (initiated as a study of perspective, in which sight perception is the guiding sense). Introduced by Desargues (1639), it is plausible that Leibniz could foresee not only a CG encompassing all those domains, but also the deep relationships between Euclidean and projective geometry that were discovered in the XIX century. But I have no factual evidence for this theoretical plausibility beyond the considerations on perspective in the report Cortese 2016, the references quoted there related to this issue, and the trust in the great generality of Leibniz's thought.

Let us also comment that the CG is not as utopic as the CU. For one thing, it refers to mathematics, where the language of logic is in principle sufficient. But it is not completely free from that unreal character, basically because it seems to implicitly assume that there is a single geometric world to be captured by 'the' CG, a scenario propounded later by Kant. As is well known, however, there are many geometric worlds and some adjustment of any CG has to possible if it is to be relevant for all those worlds. To that regard, it is useful to quote Poincaré: "I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect [...] geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics".

Unfortunately the impact that the CG could have had on the development of GA, or on other more general mathematical systems, was hindered by the fact that public notice about it had to wait until the publication of Huygens' correspondence in 1833 (by Uylenbroek). Even then, it took a while until some interest arose, and then the effect was somewhat indirect. For our concerns here, it will suffice to recall a momentous episode of the Princely Jablonowski Society Prize for the Sciences and to trace its consequences. In 1845, the prize (48 gold ducats) was for "the restoration and further development of the geometrical calculus invented by Leibniz, or the construction of one equivalent to it". The prize was awarded (in 1846, 200 years after Leibniz's birthday) to Herrmann Grassmann (1809-1877), the author of the book Grassmann 1844 (*Ausdehnungslehre*), for the work Grassmann 1847 (*Geometrisches Analyse*). It was the only one submitted and the report was published in Möbius 1847.

As it turns out, the true winner of the Jablonowsky prize was the *Ausdehnungslehre*, for its genesis and unfolding clearly show that Grassmann was on the right track for constructing a CG and thus that he needed not more than to push a bit forward his ideas to meet the prize requirements. In the words of Couturat 1901, "Grassmann took advantage of this opportunity to explain his *Extension Calculus*, and to link it to Leibniz's project, while criticizing the philosopher's rather formless essay". Indeed, in the introduction of Grassmann 1847, the author stresses that his main tool was the 'geometric analysis' he had developed, and that with further development provided a mature fulfillment of Leibniz's embryonic idea.

In retrospect, the fundamental value of the *Ausdehnungslehre* and the prize memoir, is that it shifted the focus of mathematics from the study of magnitudes to the study of (abstract) structures and relations. In particular, he essentially completed a conceptual building of linear algebra, including Grassmann's exterior algebra and the insight that quadratic forms were the natural entity to express metrical notions in linear spaces. Thus his decisive steps in the directions envisioned by Leibniz's CG were based on remarkably general mathematical ideas that have stood very well the passage of time. But the recognition of its deep significance also took many years, even after the publication of the much more readable second edition Grassmann 1862.

This is a convenient point to state the purposes of this paper in a more specific form. After the considerations so far, it makes sense to regard the *Ausdehnungslehre* as a natural and suitable ground on which to graft a GA that meets the aspirations of Leibniz's CG. Among the many important and interesting developments in this direction, our interest will be in the form that yields a comprehensive view with a rather moderate effort and we will show with a few selected examples its bearing on geometry and physics.

For those interested in the analysis of Leibniz's CG in terms of the historical documents, we refer to the text included in Leibniz 1858, or to later studies, such as Couturat 1901 (chapter 9), Leibniz 1995 (particularly the introductory study, Echeverría 1995). This text reproduces the original Latin documents on left pages and the French translations on the right ones. It is also worth mentioning Leibniz 2011 (particularly the introduction Echeverría 2011), in which a good sample of Leibniz's writings are translated onto Spanish. The masterful Antognazza 2009 is also very helpful.

4.- Development of Geometric Algebra.

William K. Clifford (1845-1879) coined the name *geometric algebra* (GA) in Clifford 1878. This landmark work advanced a synthesis of Grassmann's *Ausdehnungslehre* and Hamilton's *Quaternion algebra* (Hamilton 1843). It is worth noting that Clifford assessed Grassmann's work as follows:

> *"Until recently I was unacquainted with the* Ausdehnungslehre *[...]. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work and my conviction that its principles will exercise a vast influence upon the future of science".*

The main innovation introduced by Clifford was the *geometric product*, which is an *associative* bilinear product on the Grassmann algebra of multivectors (exterior algebra). The crucial structure discovered by Clifford, called *geometric algebra*, or *Clifford algebra* for many later authors, is what we regard as a CG in our current understanding and will be described later in this paper along the lines advanced in the foreword. Here let us just anticipate that the complex numbers and the quaternions are very special cases of geometric algebras and that all their properties appear in a natural way, with its full geometric meaning, when seen in that light.

Clifford's idea of geometric algebra was independently discovered by Rudolf Lipschitz (1832-1903) and in some important respects he went further than Clifford, particularly with the introduction of what today are called Lipschitz groups, which appear in a natural way in the analysis of how geometric algebra encodes geometric transformations (Lipschitz 1880).

Let us also take notice that at the end of the XIX century there were a number of contributions about the logical and philosophical foundations of the system of geometric algebra, and which pay ample homage to Grassmann's innovations, and indirectly to those of Leibniz. Two important works in that direction are Peano 1896 and Whitehead 1898. According to Peano, "The first step in geometrical calculus was taken by Leibniz, whose vast mind opened several new paths to mathematics", stating that that calculus "differs from Cartesian geometry in that whereas the latter operates analytically with coordinates, *the former operates directly on geometrical entities*" (emphasis added). He also recognizes Grassmann 1844 as a landmark that was "little read and not appreciated by his contemporaries, but later it was found admirable by many scientists". According to Whitehead, "The greatness of my obligations to Grassmann will be understood by those who have mastered his two *Ausdehnungslehres*. The technical development of the subject is inspired chiefly by his work of 1862, but the underlying ideas follow the work of 1844".

It is a historical pity that Clifford's or Lipschitz's syntheses had to wait over six decades for their flourishing. Clifford died in Funchal (Madeira) in 1879, aged 33, precisely two centuries after Leibniz's communication to Huygens. Then vector analysis, developed by Gibbs and Heaviside from the quaternion calculus, eclipsed those ideas. The immediate and lasting popularity of vector analysis is due to the fact that it is a no frills calculus restricted to the three-dimensional Euclidean space and that it can be used to phrase classical mechanics and electromagnetism. Other new and important developments were unaware of the value of GA and because of this, they appeared as isolated advances, and it was not until decades later that the GA symbolism brought unity and strength. The most visible cases were Minkowsky's space-time geometry (1908), Pauli's quantum theory of spin (1926), Dirac's equation for the relativistic electron (1928), Cartan's theory of spinors (1938), its algebraic presentation by Chevalley (1954), and, closer to the mark, but almost unknown at the time, Riesz (1958).

The founding document of the new era of GA is Hestenes 1966. It uncovers the full geometrical meaning of the Pauli algebra (as the GA of the Euclidean three-dimensional space), of the Dirac algebra (as the GA of the Minkowsky space), clarifying the deep relationship between the two, and finds a neat interpretation of the Pauli and Dirac spinors. In this and later works, particularly Hestenes-Sobczyk 1984, Hestenes sets up a fully developed geometric calculus which allows him to write Maxwell's equations as a single equation in the Dirac algebra (this had already been observed in Riesz 1958) and uncover a deep geometric meaning of Dirac's equation (Dirac 1928). All the references cited in the Foreword posterior to 1966 share Hestenes' vision, which quite likely Clifford would have accomplished had he not died so young. In the last years, GA has been applied or connected to a great variety of fields, including solid mechanics, robotics, electromagnetism and wave propagation, general relativity, cosmology, computer graphics, computer vision, pattern theory, molecular design, symbolic algebra, automated theorem proving, and quantum computing. See Xambó-Descamps 2018, sections 6.3 to 6.5, for a more detailed account that includes references to major published words.

5.- An elementary view of GA[2].

One of the driving ideas in mathematics has been to extend a given structure in order to include some new desirable features. The successive extensions of the notion of number provide a good illustration. Starting from the *natural numbers*, $\mathbb{N} = \{1, 2, 3, ...\}$, the inclusion of 0 and the *negative numbers* leads to the *integers*, $\mathbb{Z} = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$, in which the *difference* of any two numbers is always defined. Now division is not always possible in $\mathbb{Z}$, but one can introduce *fractions* or *rational numbers*,

$$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n{\neq}0\},$$

in which *division* by a non-zero rational number is always possible. The real numbers $\mathbb{R}$ are the natural extension of $\mathbb{Q}$ that makes possible to take the *upper bound of bounded* sets, and $\mathbb{C}$ is the natural extension of $\mathbb{R}$ in which *negative real numbers have roots*, in particular, $i = \sqrt{-1}$.

Similarly, GA arises out of the desire to multiply vectors with the usual rules of multiplying numbers, including the usual rules for taking inverses.

Geometric algebra of the Euclidean plane. For definiteness, let us start with an Euclidean plane, E_2.[3] So E_2 is a real vector space of dimension 2 endowed with an *Euclidean metric* q (a *symmetric bilinear* map)

$$q : E_2 \times E_2 \to E_2$$

such that $q(a,a) > 0$ for any $a \neq 0$).[4] We want to *enlarge* E_2 to a system $\mathcal{G}_2$ in which vectors can be multiplied, and non-zero vectors inverted, with the usual rules. To elicit the expected goods that $\mathcal{G}_2$ is going to bring us, let us first make a few remarks. By xy we denote the product of $x, y \in \mathcal{G}_2$ (simple juxtaposition of the factors) and we say that it is the *geometric product* of x and y.

Technically, the structure of $\mathcal{G}_2$ with the geometric product is supposed to be an *associative* and *unital* $\mathbb{R}$-algebra. This means that $\mathcal{G}_2$ is a real vector space and that the geometric product is bilinear, associative, with unit $1 \in \mathcal{G}_2$. The

2 This section is an adaptation to the present context, often with different notations and in an abridged form, of some parts of chapters 1 and 3 of the book Xambó-Decamps, Lavor, Zaplana 2018. A more general and systematic account, particularly from the point of view of mathematical foundations, can be found in Xambó-Decamps 2018.

3 Many authors take $E_2 = \mathbb{R}^2$. Our notation is meant to stress that no basis is special.

4 Instead of $q(a,b)$, we will also write $a \cdot b$. As we will see, however, the two notations usually mean different things when a or b are not vectors.

map $\mathbb{R} \to \mathcal{G}_2$, $\lambda \mapsto \lambda 1$, allows us to regard $\mathbb{R}$ as embedded in $\mathcal{G}_2$, and so we will not distinguish between $\lambda \in \mathbb{R}$ and $\lambda 1 \in \mathcal{G}_2$. Such elements are the *scalars* of $\mathcal{G}_2$. We also have $E_2 \subset \mathcal{G}_2$, and its elements are the *vectors* of $\mathcal{G}_2$. We assume that $\mathbb{R} \cap E_2 = \{0\}$.

Let $a \in E_2$. If a is to have an inverse a' with respect to the geometric product, it is natural to assume that $a' \in \langle a \rangle = \{\lambda a : \lambda \in \mathbb{R}\}$, because $\langle a \rangle$ is the only subset of E_2 that can be constructed out of a using the linear structure. But then $1 = a'a = (\lambda a)a = \lambda a^2$, which implies that a^2 *must be a non-zero scalar*. If now $a, b \in E_2$, then

$$(a+b)^2 = a^2 + b^2 + ab + ba \Rightarrow ab + ba \in \mathbb{R}.$$

The main insight of Clifford was to postulate that

$$ab + ba = q(a,b) + q(b,a) = 2q(a,b),$$

because q is symmetric, and in particular that

$$a^2 = q(a),$$

where for simplicity we write $q(a) = q(a,a)$. In any case, the expression

$$ab + ba$$

defines a symmetric bilinear form of E_2, and if $a \neq 0$, then $a^{-1} = a/q(a)$ is the inverse of $a \in E_2$. Notice also that $ab = -ba$ if and only if $q(a,b) = 0$, that is, if and only if a and b are *orthogonal*.

To go further, let us take an arbitrary *orthonormal basis* $u_1, u_2 \in E_2$,[5] which means that

$$q(u_1) = q(u_2) = 1 \text{ and } q(u_1,u_2) = 0 \Leftrightarrow u_1^2 = u_2^2 = 1 \text{ and } u_2u_1 = -u_1u_2.$$

These rules imply, owing to the bilinearity of the geometric product, that all geometric products of (any number of) vectors belong to the subspace

$$\langle 1, u_1, u_2, u_1u_2 \rangle \subseteq \mathcal{G}_2.$$

Since we want that $\mathcal{G}_2$ be a minimal solution to our problem, it is thus natural to assume that

$$\langle 1, u_1, u_2, u_1u_2 \rangle = \mathcal{G}_2.$$

Now it is important to insure that $1, u_1, u_2, u_1u_2$ are necessarily linearly independent. Indeed, if

$$\lambda + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_{12} u_1 u_2 = 0$$

5 Grassmann, and many others ever since, use the notation $e_1, e_2, \ldots$ Here we avoid it to prevent potential confusions in important expressions involving exponentials.

is a linear relation, then we can multiply by u_1 from the left and from the right and we obtain

$$\lambda + \lambda_1 u_1 - \lambda_2 u_2 - \lambda_{12} u_1 u_2 = 0.$$

Adding the two last relations, we conclude that $\lambda + \lambda_1 u_1 = 0$ and hence

$$\lambda = \lambda_1 = 0.$$

So we are left with $\lambda_2 u_2 + \lambda_{12} u_1 u_2 = 0$, and this relation leads, after multiplying on the right by u_2, to $\lambda_2 + \lambda_{12} u_1 = 0$, and hence to $\lambda_2 = \lambda_{12} = 0$ as well. This proves the claim.[6]

Consequently, we have a decomposition $\mathcal{G}_2 = \mathcal{G}_2^0 + \mathcal{G}_2^1 + \mathcal{G}_2^2$, with

$$\mathcal{G}_2^0 = \langle 1 \rangle = \mathbb{R},\ \ \mathcal{G}_2^1 = \langle u_1, u_2 \rangle = E_2, \text{ and } \mathcal{G}_2^2 = \langle u_1 u_2 \rangle,$$

and it is not hard to see that it is independent of the orthonormal basis used **N1** (to ease the reading we collect some of the mathematical deductions with labels **N1**, … in the Notes section at the end). The elements of $\mathcal{G}_2^2$ are called *bivectors* or *pseudoscalars*. To get a better appreciation of this component of $\mathcal{G}_2$, consider the map $E_2^2 \to \mathcal{G}_2$, $(a, b) \mapsto \frac{1}{2}(ab - ba)$. Since this map is bilinear and skew-symmetric, it gives a linear map $\Lambda^2 E_2 \to \mathcal{G}_2$ such that

$$a \wedge b \mapsto \tfrac{1}{2}(ab - ba).$$ **N2**

In particular, $u_1 \wedge u_2 \mapsto u_1 u_2$, and so we have a canonical linear isomorphism $\Lambda^2 E_2 \simeq \mathcal{G}_2^2$. Since $\Lambda^0 E_2 = \mathbb{R} = \mathcal{G}_2^0$ and $\Lambda^1 E_2 = E_2 = \mathcal{G}_2^1$, we actually have a canonical *linear* isomorphism

$$\Lambda E = \Lambda^0 E_2 + \Lambda^1 E_2 + \Lambda^2 E_2 \simeq \mathcal{G}_2^0 + \mathcal{G}_2^1 + \mathcal{G}_2^2 = \mathcal{G}_2.$$

This allows us to copy onto $\mathcal{G}_2$ the exterior product of ΛE. The result is the *exterior product* in $\mathcal{G}_2$ coexisting, from now on, with the geometric product. The basic relation between the two products is the following key relation, also discovered by Clifford:

$$ab = \tfrac{1}{2}(ab + ba) + \tfrac{1}{2}(ab - ba) = a \cdot b + a \wedge b.$$

From the geometric point of view, the elements of $\mathcal{G}_2^2 = \Lambda^2 E_2$ represent, following the standard interpretation of the Grassmann algebra, *oriented areas*.

Among these, the *unit area* $\boldsymbol{i} = u_1 \wedge u_2 = u_1 u_2$ plays a very important role. It anticommutes with vectors and satisfies $\boldsymbol{i}^2 = u_1 u_2 u_1 u_2 = -u_1^2 u_2^2 = -1$, so that $\boldsymbol{i}$ is *a geometric root of* -1. This entails several more marvels. If we set

$$\mathcal{G}_2^+ = \mathcal{G}_2^0 + \mathcal{G}_2^2 = \langle 1, \boldsymbol{i} \rangle,$$

6 This proof is a particular case of the general method introduced in Riesz 1958 (*Riesz method* in this paper).

which is called the *even geometric algebra*, we see that $\mathcal{G}_2^+ \simeq \mathbb{C}$, via $\alpha+\beta\boldsymbol{i} \mapsto \alpha+\beta i$, but of course the geometric meaning of the left hand side (which henceforth will be denoted $\mathbf{C}$ and called *complex scalars*) is lost when we move to i, the formal square root of -1. On the other hand, the map $\mathbb{R} = \mathcal{G}_2^0 \to \mathcal{G}_2^2$, $\lambda \mapsto \lambda\boldsymbol{i}$, is a linear isomorphism, with inverse $\mathcal{G}_2^2 \to \mathcal{G}_2^0 = \mathbb{R}$, $s \mapsto -s\boldsymbol{i}$ (since $\boldsymbol{i}^2 = -1$). Because of this, we say that the area elements are the *pseudoscalars*, and in particular that $\boldsymbol{i}$ is the *unit pseudoscalar* (associated to the basis u_1, u_2). If we change the orthonormal basis, we have seen that the corresponding pseudoscalar is $\pm\boldsymbol{i}$. Still another marvel is that $\mathcal{G}_2^1 = E_2$ is a $\mathbf{C}$-vector space, because $u_1\boldsymbol{i} = u_2$ and $u_2\boldsymbol{i} = -u_1$. Note that these relations show that the linear isomorphism $E_2 \to E_2$, $a \mapsto a\boldsymbol{i}$, is the counterclockwise rotation by $\pi/2$. More generally, the map

$$E_2 \to E_2,\ a \mapsto ae^{\theta\boldsymbol{i}},$$

is the rotation of a by θ in the $\boldsymbol{i}$ orientation. Here $e^{\theta\boldsymbol{i}}$ expands, as usual, to $\cos\theta + \boldsymbol{i}\sin\theta$, and so

$$u_1e^{\theta\boldsymbol{i}} = u_1\cos\theta + u_2\sin\theta, \quad u_2e^{\theta\boldsymbol{i}} = u_2\cos\theta - u_1\sin\theta \quad \text{(see Figure 1).}$$ [N3]

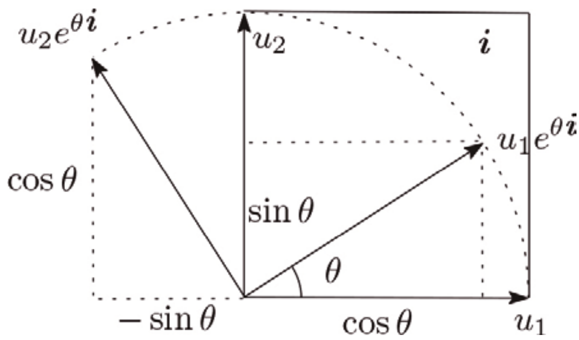


Figure 1. u_1, u_2, $\boldsymbol{i}$ and the action of $e^{\theta\boldsymbol{i}}$ on them.

Geometric algebra of the Euclidean space. To fully appreciate GA, even in the case of dimension 2, we need to explore the geometric algebra $\mathcal{G}_3$ of E_3, the three dimensional Euclidean space. The development is similar to that of $\mathcal{G}_2$ and we only need to summarize the main points. In this case we have the decomposition

$\mathcal{G}_3 = \mathcal{G}_3^0 + \mathcal{G}_3^1 + \mathcal{G}_3^2 + \mathcal{G}_3^3$, with

$\mathcal{G}_3^0 = \langle 1 \rangle = \mathbb{R}$, $\mathcal{G}_3^1 = \langle u_1, u_2, u_3 \rangle = E_3$, $\mathcal{G}_3^2 = \langle u_1u_2, u_1u_3, u_2u_3 \rangle$, $\mathcal{G}_3^3 = \langle u_1, u_2, u_3 \rangle$,

where u_1, u_2, $u_3 \in E_3$ is an orthonormal basis. The decomposition does not depend on the basis used, as follows from the fact that there is a *canonical* linear isomorphism $\Lambda^k E \simeq \mathcal{G}_3^k$ for $k = 0, 1, 2, 3$. For $k = 0, 1$, it is the identity. For $k = 2$ it is defined as for E_2:

$a \wedge b \mapsto \frac{1}{2}(ab - ba)$.

And for $k = 3$,

$a \wedge b \wedge c \mapsto \frac{1}{6}(abc + bca + cab - acb - bac - cba)$.

So we get an exterior product in $\mathcal{G}_3$ by copying the exterior product of ΛE via the described isomorphisms. Notice that we have

$u_i \wedge u_j = u_iu_j$ and $u_i \wedge u_j \wedge u_k = u_iu_ju_k$.[N4]

The pseudoscalar $\mathbf{i} = u_1u_2u_3$ (the *unit volume*) plays a special role: it commutes with all vectors and $\mathbf{i}^2 = -1$. Thus we have a linear isomorphism

$\mathcal{G}_3^1 \simeq \mathcal{G}_3^2$, $a \mapsto a\mathbf{i}$,

where $u_1, u_2, u_3 \in \mathcal{G}_3^1$ are mapped to the basis

$u_1^* = u_2u_3$, $u_2^* = u_3u_1$, $u_3^* = u_1u_2$ of $\mathcal{G}_3^2$,

with inverse $\mathcal{G}_3^2 \simeq \mathcal{G}_3^1$ given by $b \mapsto -b\mathbf{i}$. And of course there is a linear isomorphism $\mathcal{G}_3^0 \simeq \mathcal{G}_3^3$, $\lambda \mapsto \lambda\mathbf{i}$, with inverse $s \mapsto -s\mathbf{i}$. In general, we have an isomorphism $\mathcal{G}_3^k \simeq \mathcal{G}_3^{3-k}$, $x \mapsto x^* = x\mathbf{i}$, which is usually called *Hodge duality* (see Figure 2).

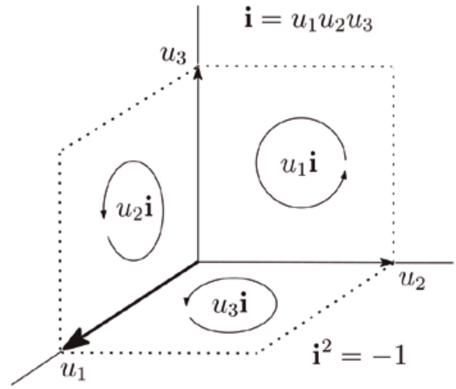
Scalars	1	
Vectors: oriented segments (*polar vectors*)	u_1, u_2, u_3	
Bivectors: oriented areas (*axial vectors*)	u_1^*, u_2^*, u_3^*	
Pseudoscalars: oriented volumes	$\mathbf{i}$	
Hodge duality	$u_k^* = u_k\mathbf{i}$	

Figure 2. Geometric algebra units of E_3.

Rotations and rotors. We are now ready to describe rotations of E_3 in terms of $\mathcal{G}_3$ and use it to establish some interesting consequences. Let n be a unit vector (that is, $n^2 = q(n) = 1$). Then it is easy to see that $n\mathbf{i}$ is the unit area of the perpendicular plane $n^\perp$ and hence $a \mapsto ae^{\theta n\mathbf{i}}$ rotates any $a \in n^\perp$ by θ in the sense of $n\mathbf{i}$. The important observation now is that the rotation of any vector $a \in E_3$ by θ about the axis n is given by what may be called *Euler's spinor formula:*

$$a \mapsto e^{-\theta n\mathbf{i}/2}\, a\, e^{\theta n\mathbf{i}/2}.$$

To prove this, it is enough to show that $a = n$ is fixed and that if a is orthogonal to n, then it is rotated by θ in $n^\perp$. Indeed, since n commutes with $e^{-\theta n\mathbf{i}/2}$, the value of the formula for $a = n$ is $ne^{-\theta n\mathbf{i}/2}e^{\theta n\mathbf{i}/2} = n$; and if $a \in n^\perp$, then a anticommutes with n, hence also with $e^{-\theta n\mathbf{i}/2}$, and the formula yields $ae^{\theta n\mathbf{i}}$, which is, as we have remarked, the rotation of a in the plane $n^\perp$ by θ in the sense of $n\mathbf{i}$.

Let us say that $R_{n,\theta} = e^{\theta n\mathbf{i}/2}$ is the *rotor* of the rotation $\rho_{n,\theta}$ about n by θ. Thus

$$\rho_{n,\theta}(a) = R^{1}_{n,\theta}\, a\, R_{n,\theta}.$$

If $\rho_{n',\theta'}$ is another rotation, and $R_{n',\theta'}$ its rotor, then the product $R_{n,\theta}\, R_{n',\theta'}$ is clearly the rotor $R_{n'',\theta''}$ of the composition $\rho_{n',\theta'}\, \rho_{n,\theta}$. Therefore we have the equation

$$\cos\theta''/2 + n''\,\mathbf{i}\sin\theta''/2 = (\cos\theta/2 + n\mathbf{i}\sin\theta/2)\,(\cos\theta'/2 + n'\mathbf{i}\sin\theta'/2)$$

which itself is equivalent (equating scalar and bivector parts) to the equations

$$\cos\theta''/2 = \cos\theta/2\ \cos\theta'/2 - (n\cdot n')\sin\theta/2\sin\theta'/2$$

$$n''\sin\theta''/2 = n\ \sin\theta/2\cos\theta'/2 + n'\cos\theta/2\ \sin\theta'/2 + (n\times n')\sin\theta\cos\theta''/2$$

where $n \times n' = (n \wedge n')\mathbf{i}$ is the *cross product* (the dual of the wedge product). These are the famous Olinde Rodrigues's formulas obtained in 1843 with a remarkably long and involved computation using Cartesian coordinates.

The algebra $\mathcal{G}_3$ is the main tool used, for the first time, in the excellent treatise Hestenes 1990 to formulate classical mechanics. Aside from the many advantages that this approach provides, including the use of rotors as dynamical variables, $\mathcal{G}_3$ plays a crucial role, as we will see, as the relative algebra of an observer in the geometric algebra approach to special relativity (space-time algebra).

Quaternions. $\mathcal{G}_3$ provides a geometric realization $\mathbf{H}$ of Hamilton's quaternions $\mathbb{H}$. Let

$$\boldsymbol{i}_1 = u_1\mathbf{i},\quad \boldsymbol{i}_2 = u_2\mathbf{i},\quad \boldsymbol{i}_3 = u_3\mathbf{i}.$$

Then we have $\mathcal{G}_3^+ = \langle 1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\rangle$, which we will denote $\mathbf{H}$. This even geometric algebra is isomorphic to $\mathbb{H}$, because the unit areas $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ satisfy Hamilton's famous relations:[N5]

$$\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = -1,\quad \boldsymbol{i}_1\boldsymbol{i}_2 = -\boldsymbol{i}_2\boldsymbol{i}_1 = \boldsymbol{i}_3,\quad \boldsymbol{i}_2\boldsymbol{i}_3 = -\boldsymbol{i}_3\boldsymbol{i}_2 = \boldsymbol{i}_1,\quad \boldsymbol{i}_3\boldsymbol{i}_1 = -\boldsymbol{i}_1\boldsymbol{i}_3 = \boldsymbol{i}_2.$$

The advantage of $\mathbf{H}$ over $\mathbb{H}$ is the direct relation to the geometry of E_3. For example, in Euler's spinor formula, $n\mathbf{i} = n_1\boldsymbol{i}_1 + n_2\boldsymbol{i}_2 + n_3\boldsymbol{i}_3$ is a bivector (a pure quaternion in the habitual terminology) and Euler's spinor formula coincides with the way quaternions are used to produce rotations. Note that the rotation given by the rotor $\boldsymbol{i}_k = e^{\pi\boldsymbol{i}_k/2} = e^{\pi u_k\mathbf{i}/2}$ is $\rho_{u_k,\pi}$, that is, the axial symmetry about u_k.

Geometric covariance. Another advantage of $\mathcal{G}_3$ is that Euler's spinor formula can be applied to any multivector. Actually the map

$$\mathcal{G}_3 \to \mathcal{G}_3,\quad x \mapsto x' = e^{-\theta n\mathbf{i}/2}\, x\, e^{\theta n\mathbf{i}/2}$$

is an automorphism of $\mathcal{G}_3$ (if $R = e^{\theta n\mathbf{i}/2}$, $(xy)' = R^{-1}(xy)R = R^{-1}xRR^{-1}yR = x'y'$). This is what may be called the principle of *geometric covariance*. The simplest illustration is that

$$(a \wedge b)' = a' \wedge b',$$

as $(a \wedge b)' = \frac{1}{2}(ab - ba)' = \frac{1}{2}(a'b' - b'a') = a' \wedge b'$. Now notice that the right hand side constructs the oriented area over the rotated vectors a' and b', which is the rotated oriented area, while the left hands side gives the good news that we can arrive at the same result by directly 'rotating' the oriented area expressed as a bivector.

Here is another illuminating example. Let us apply the transformation to a rotor $S = e^{\alpha p \mathbf{i}/2}$ (p a unit vector, α real), giving $S' = R^{-1}SR$. Now the geometric meaning of S' is easy to ascertain: from

$$S' = (\cos \alpha/2 + p\mathbf{i} \sin \alpha/2)' = \cos \alpha/2 + p'\mathbf{i} \sin \alpha/2$$

we see that S' is the rotor of the rotation about p', which is the rotation of p about n by θ. In other words, to 'rotate' a rotation $\rho_{p,\alpha}$ *to* $\rho_{p',\alpha}$, it is enough to 'rotate' the rotor S to S', for S' is the rotor of $\rho_{p',\alpha}$.

The Pauli representation. The algebra $\mathcal{G}_3$ is isomorphic to Pauli's spin algebra. In technical terms, the latter is a matrix representation of the former. In this representation, u_1, u_2, u_3 are mapped to Pauli's matrices

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\ \ \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix},\ \ \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

which satisfy Clifford's relations

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{j,k} \quad (\delta_{j,k} \text{ is } 0 \text{ if } j \neq k \text{ and } 1 \text{ if } j = k).$$

The Pauli matrices belong to $\mathbb{C}(2)$, the 2×2 complex matrices, and it is easy to check that $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3, \sigma_1\sigma_2\sigma_3$ form a basis of $\mathbb{C}(2)$ as a real vector space (σ_0 denotes the identity matrix). We conclude that we have an isomorphism of real algebras $\mathcal{G}_3 \simeq \mathbb{C}(2)$. Note, however, that the rich geometric structure of $\mathcal{G}_3$ is invisible in the algebra $\mathbb{C}(2)$.[7]

The indices in the Pauli matrices are explained as follows. Those matrices have eigenvalues ±1, and the corresponding eigenvectors in $\mathbb{C}^2$ (this is called *Pauli's spinor space*) are [1, ±1], [1, ±i], and [1,0] and [0,1], respectively. These eigenvalues become the unit points on the axes u_1, u_2, u_3, under the *spinor map* $S^3 \to S^2$ (*Hopf fibration* in mathematics texts), where S^3 is the unit sphere in $\mathbb{C}^2 \simeq \mathbb{R}^4$ ($[\xi_0, \xi_1] \in S^3$ (if and only if $\xi_0\bar{\xi}_0+\xi_1\bar{\xi}_1=1$) and S^2 is the unit sphere in $E_3 \simeq \mathbb{R}^3$ ($a = a_1u_1 + a_2u_2 + a_3u_3 \in S^2$ if and only if $a_1^2 + a_2^2 + a_3^2 = 1$). By the spinor

7 In the presentation of $\mathcal{G}_3$, many authors use the symbols $\sigma_1, \sigma_2, \sigma_3$ (or $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$) instead of u_1, u_2, u_3. For reasons that we shall explain, we will revert to this convention in the presentation of the space-time algebra below. Notice also that the Pauli representation provides a proof of the existence of $\mathcal{G}_3$.

map, the image $a \in S^2$ of $\psi = [\xi_0, \xi_1]$ is given by

$$a_1 = \xi_0\bar{\xi}_1 + \bar{\xi}_0\bar{\xi}_1, \quad a_2 = i(\xi_0\bar{\xi}_1 - \bar{\xi}_0\xi_1), \quad a_3 = \xi_1\bar{\xi}_1 - \xi_0\bar{\xi}_0.$$

With this, the claim follows immediately.

6.- Space-time algebra.

The final example we will examine is the geometric algebra $\mathcal{D} = \mathcal{G}_{1,3}$ of the Minkowski space-time $E_{1,3}$ (the *Dirac algebra*).[8] The metric of $E_{1,3}$ will be denoted η (instead of q). It has signature (1,3), which means that there exist bases $\boldsymbol{\gamma} = \gamma_0, \gamma_1, \gamma_2, \gamma_3$ of $E_{1,3}$ such that $\eta(\gamma_\mu, \gamma_\nu) = \eta_{\mu\nu}$, where

$$\eta_{\mu\nu} = 0 \text{ for } \mu \neq \nu, \quad \eta_{00} = 1, \quad \eta_{kk} = -1 \text{ for } k = 1, 2, 3.$$[9]

Such bases $\boldsymbol{\gamma}$, which are said to be *orthonormal* in mathematics, are called *inertial frames* in special relativity texts. The axis γ_0 is the *temporal axis* and $\gamma_1, \gamma_2, \gamma_3$ the *spatial axes*. In general, a vector a is *timelike* (*spacelike*) if $\eta(a) > 0$ ($\eta(a) < 0$). Vectors such that $\eta(a) = 0$ are called *null vectors* (*isotropic* in mathematics). Figure 3 summarizes the correspondence between a short list of mathematics and special relativity terms. Henceforth we will not bother about such language differences.

Symbols	Mathematics	Special relativity
$a \in E_{1,3}$	Vectors	Events or vectors
$\eta(a) > 0$	Positive vector	Time-like vector
$\eta(a) < 0$	Negative vector	Space-like vector
$\eta(a) = 0$	Isotropic vector	Null or light-like vector
$\eta(La) = \eta(a)$, L linear	Isometry	Lorentz transformation
$\eta(\gamma_\mu, \gamma_\nu) = \eta_{\mu\nu}$	Orthonormal basis	Inertial frame

Figure 3. Translating between symbolism, mathematics, and physics.

Now we proceed to construct the geometric algebra $\mathcal{D} = \mathcal{G}_{1,3}$ of $E_{1,3}$ by analogy with the case $\mathcal{G}_3$. Starting with a frame $\boldsymbol{\gamma}$, let us write $\gamma_J = \gamma_{j_1} \cdots \gamma_{j_k}$ for

8 The detailed study of this algebra in Hestenes 1966 was the harbinger of the contemporary understanding of geometric algebra and its applications.

9 As we will see, the symbols γ_μ correspond to the so-called Dirac-matrices much in the same way as the reference symbols for $\mathcal{G}_3$ correspond to Pauli's σ-matrices.

$J = j_1, \cdots, j_k \in N = \{0,1,2,3,4\}$. Then we find that the 2^4 products γ_J with $j_1 < \cdots < j_k$, $k = 0, \cdots, 4$, form a basis of $\mathcal{D}$ and that by grouping the products for the different k we have a grading

$$\mathcal{D} = \mathcal{D}^0 + \mathcal{D}^1 + \mathcal{D}^2 + \mathcal{D}^3 + \mathcal{D}^4.$$

Thus $\mathcal{D}^0 = \langle 1 \rangle = \mathbb{R}$ (as $\gamma_\emptyset = 1$) and $\mathcal{D}^1 = \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \rangle = E_{1,3}$. The other terms can be conveniently described by introducing the bivectors

$$\sigma_k = \gamma_k\gamma_0 = \gamma_{k0} \ (k = 1,2,3),$$

the pseudoescalar $\mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_{0123}$ and the notation $x^* = x\mathbf{i}$ (*Hodge dual of x*). Then

$$\sigma_1^* = -\gamma_2\gamma_3 = -\gamma_{23}, \quad \sigma_2^* = -\gamma_3\gamma_1 = -\gamma_{31}, \quad \sigma_3^* = -\gamma_1\gamma_2 = -\gamma_{12}$$

and

$$\mathcal{D}^2 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_1^*, \sigma_2^*, \sigma_3^* \rangle.$$

Similarly $\gamma_0^* = \gamma_{123}, \gamma_1^* = \gamma_{023}, \gamma_2^* = \gamma_{031}, \gamma_3^* = \gamma_{012}$, and

$$\mathcal{D}^3 = \langle \gamma_0^*, \gamma_1^*, \gamma_2^*, \gamma_3^* \rangle.$$

Finally it is clear that $\mathcal{D}^4 = \langle \mathbf{i} \rangle = \langle 1^* \rangle$. Figure 4 summarizes all these relations at a glance.

<table>
<tr><th>Grade</th><th>Names</th><th>Bases</th><th>Notations</th></tr>
<tr><td>0</td><td>Scalars</td><td>1</td><td rowspan="5">$\mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3$
$x^* = x\mathbf{i}$
$\eta(x) = 1$
$\eta(\bar{x}\ ) = -1$
$\mathbf{i}^2 = -1$</td></tr>
<tr><td>1</td><td>Vectors</td><td>$\gamma_0 \quad \bar{\gamma}_1 \quad \bar{\gamma}_2 \quad \bar{\gamma}_3$</td></tr>
<tr><td>2</td><td>Bivectors</td><td>$\bar{\sigma}_1 \quad \bar{\sigma}_2 \quad \bar{\sigma}_3 \qquad \sigma_1^* \quad \sigma_2^* \quad \sigma_3^*$</td></tr>
<tr><td>3</td><td>Pseudovectors</td><td>$\bar{\gamma}_0^* \quad \gamma_1^* \quad \gamma_2^* \quad \gamma_3^*$</td></tr>
<tr><td>4</td><td>Pseudoscalars</td><td>$\bar{1}^*$</td></tr>
</table>

Figure 4. Basis of $\mathcal{D}$. Products for which η is -1 are distinguished with an overbar, and otherwise $\eta(x) = +1$. Note that if x is a basis element, then $x^2 = \eta(x)$ only happens for scalars, vectors and pseudoscalars. For bivectors and pseudovectors, $x^2 = -\eta(x)$.

We still have canonical linear isomorphisms $\Lambda^k E_{1,3} \simeq \mathcal{D}^k$, and hence a canonical linear isomorphism $\Lambda E_{1,3} \simeq \mathcal{D}$, that allows us to copy the exterior product of $\Lambda E_{1,3}$ to an exterior product of $\mathcal{D}$. In particular, the metric η can be extended canonically to a metric of ΛE [N6] and hence to a metric of $\mathcal{D}$. For the basis elements the value of η is just the product of the η values of the factors. For example, $\eta(\sigma_k) = \eta(\gamma_k\gamma_0) = \eta(\gamma_k)\eta(\gamma_0) = -1$, $\eta(\sigma_k^*) = (-1)(-1) = 1$, and

$\eta(\mathbf{i}) = (+1)(-1)(-1)(-1) = -1$. The scheme also reflects the Hodge dualities $\mathcal{D}^k \simeq \mathcal{D}^{4-k}$, $x \mapsto x^*$ (they are anti-isometries: $\eta(x^*) = \eta(x\mathbf{i}) = -\eta(x)$).

Relative space. The space $\mathcal{E} = \langle\sigma_1, \sigma_2, \sigma_3\rangle$, which depends on the frame $\boldsymbol{\gamma}$, is called the *relative space*. It generates the even subalgebra $\mathcal{P} = \mathcal{D}^+$ and since

$$\sigma_k^2 = -\eta(\sigma_k) = 1,$$

it turns out that $\mathcal{P}$ is isomorphic to the Pauli algebra. The pseudoscalar of $\mathcal{P}$ coincides with $\mathbf{i}$, for

$$\sigma_1\sigma_2\sigma_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3 = \mathbf{i}.$$

The grading $\mathcal{P} = \mathcal{P}^0 + \mathcal{P}^1 + \mathcal{P}^2 + \mathcal{P}^3$ is easy to describe in terms of $\mathcal{D}$:

$$\mathcal{P}^0 = \mathbb{R}, \quad \mathcal{P}^1 = \mathcal{E}, \quad \mathcal{P}^2 = \mathcal{E}\mathbf{i}, \quad \mathcal{P}^3 = \langle\mathbf{i}\rangle = \mathcal{D}^4.$$

Notice that $\mathcal{P}^1 + \mathcal{P}^2 = \mathcal{E} + \mathcal{E}\mathbf{i} = \mathcal{D}^2$.

If $a \in E_{1,3}$ and we write $a = a^0\gamma_0 + a^1\gamma_1 + a^2\gamma_2 + a^3\gamma_3$, it is customary to also use the symbols t, x, y, z such that $a^0 = ct,\ \ x = a^1,\ \ y = a^2,\ \ z = a^3$ (or $a^0 = t$ if units are chosen so that c, the speed of light in empty space, is 1). The parameter t is the *time* assigned to a in the frame $\boldsymbol{\gamma}$, (x, y, z) are the *space coordinates*, and t, x, y, z are often referred to as *lab coordinates*. In these coordinates, $\eta(a)$ agrees with the familiar *Lorentz quadratic form*:

$$\eta(a) = c^2t^2 - (x^2 + y^2 + z^2).$$

The time coordinate is $t = \eta(a,\gamma_0) = a\cdot\gamma_0$. On the other hand, it defines the relative vector $\boldsymbol{a} = a \wedge \gamma_0 = x\sigma_1 + y\sigma_2 + z\sigma_3 \in \mathcal{E}$, where x, y, z, are the space coordinates of $\boldsymbol{a}$. Then $\boldsymbol{a}^2 = x^2 + y^2 + z^2$ and therefore

$$\eta(a) = t^2 - (x^2 + y^2 + z^2) = t^2 - \boldsymbol{a}^2.$$

Remark that if we let q denote the Euclidean metric of $\mathcal{E}$ such that $\sigma_1, \sigma_2, \sigma_3$ is orthonormal, then $q(\sigma_k) = -\eta(\sigma_k)$.

Inner product. Now we need to introduce the *inner product* $x\cdot y$ of two multivectors $x, y \in \mathcal{D}$. Since we want it to be bilinear, we only need to define it for basis elements, say $\gamma_J \cdot \gamma_K$. To that end, first notice that

$$\gamma_J\gamma_K = (-1)^{\iota(J,K)}\,\eta(\gamma_{J\cap K})\,\gamma_{J\Delta K} \quad (\textit{Artin's formula}),$$

where $\iota(J,K)$ denotes the number of inversions in the joint sequence J,K and $J \,\Delta\, K$ is the symmetric difference of J and K (rearranged in increasing order). For example, it is clear that $\gamma_1\gamma_{013} = \gamma_1\gamma_0\gamma_1\gamma_3 = -\gamma_1^2\gamma_{03} = \gamma_{03}$, while $\iota(1,013) = 1$ (hence $(-1)^{\iota(1,013)} = -1$), $\eta(\gamma_{1\cap 013}) = \eta(\gamma_1) = \gamma_1^2 = -1$ and $\gamma_{1\,\Delta\,013} = \gamma_{03}$.[N7]

Thus the grade of $\gamma_J\gamma_K$ is $|J\Delta K| = |J| + |K| - 2|J\cap K|$. Fixing $r = |J|$ and $s = |K|$, we see that this grade has the form $r + s - 2l$, where $l = |J\cap K|$. Consequently, the maximum possible grade is $r + s$, which occurs precisely when $J \cap K = \emptyset$, and then $\gamma_J\gamma_K = \gamma_J \wedge \gamma_K$. Similarly, the minimum possible grade is $|r-s|$ and occurs precisely when either $J \subseteq K$ (grade $s-r$) or $K \subseteq J$ (grade $r-s$). Now we can proceed to the definition of the *inner* (or *interior*) *product* $\gamma_J \cdot \gamma_K$. If $J \nsubseteq K$ or $K \subsetneq J$, or if $r = 0$ or $s = 0$, set $\gamma_J \cdot \gamma_K = 0$ (the reason for the latter rule will be seen in a moment). Otherwise (so $r, s \geq 1$ and either $J \subseteq K$ or the $K \subseteq J$), set

$$\gamma_J \cdot \gamma_K = \gamma_J\,\gamma_K = (-1)^{\iota(J,K)}\,\eta(\gamma_J)\,\gamma_{K-J} \text{ if } J \subseteq K, \text{ and } = (-1)^{\iota(J,K)}\,\eta(\gamma_K)\,\gamma_{J-K} \text{ if } K \subseteq J.$$

In particular, $\gamma_J \cdot \gamma_J = \gamma_J^2$ *if* $J \neq \emptyset$.

Key formulas. If a is a vector and x a multivector, then

$$ax = a\cdot x + a \wedge x \text{ and } xa = x\cdot a + x \wedge a.$$

This is true when x is a scalar, because in that case the interior product vanishes and the exterior product agrees with the product. By bilinearity, we can assume that $a = \gamma_j$, $x = \gamma_K$, $|K| \geq 1$. If $j \in K$, then $a \wedge x = x \wedge a = 0$, while $a \cdot x = ax$ and $x \cdot a = xa$. And if $j \notin K$, then $a \cdot x = x \cdot a = 0$, while $ax = a \wedge x$ and $xa = x \wedge a$.

Involutions. The linear involution $\mathcal{D} \to \mathcal{D}$, $x \mapsto \hat{x}$, where $\hat{x} = (-1)^r x$ for $x \in \mathcal{D}^r$, turns out to be an automorphism of $\mathcal{D}$ (*parity involution*), in the sense that

$$\widehat{xy} = \hat{x}\hat{y},\quad \widehat{x \wedge y} = \hat{x} \wedge \hat{y},\quad \widehat{x\cdot y} = \hat{x}\cdot\hat{y}$$

Similarly, the linear involution $\mathcal{D} \to \mathcal{D}$, $x \mapsto \tilde{x}$, where $\tilde{x} = (-1)^{\binom{r}{2}} x$, is an anti-automorphism of $\mathcal{D}$ (*reverse involution*), in the sense that

$$\widetilde{xy} = \tilde{y}\tilde{x},\quad \widetilde{x \wedge y} = \tilde{y} \wedge \tilde{x} \text{ and } \widetilde{x \cdot y} = \tilde{y} \cdot \tilde{x}$$

For both assertions it is enough to check the identities for two basis elements, $x = \gamma_J$ and $y = \gamma_K$, say of grades r and s, respectively. For the parity involution, note that the grades of $\gamma_J\gamma_K$, $\gamma_J \wedge \gamma_K$, and $\gamma_J \cdot \gamma_K$ are $r + s - 2l$ (l the cardinal *of* $J \cap K$), $r + s$ and $|r - s|$, respectively, and that all are congruent to $r + s$ mod 2. In the case of the reverse involution, the argument is similar if we take into account that $\tilde{\gamma}_J = \gamma_{\tilde{J}}$, where $\tilde{J}$ the reversal of the sequence J (note that reordering $\tilde{J}$ involves $\binom{r}{2}$ sign changes).

We will need the following property: an $x \in \mathcal{D}$ is a vector if and only if

$$\hat{x} = -x \text{ and } \tilde{x} = x.$$

Indeed, the first equality says that x can have only odd components ($x = x_1 + x_3$)

and the second equality says that $x_3 = 0$, because $\tilde{x} = x_1 - x_3$.

Complex structure. The *complex scalars* of $\mathcal{P}$, $\mathbf{C} = \langle 1, \mathbf{i}\rangle = \mathcal{P}^0 + \mathcal{P}^3$, coincide with the complex scalars of $\mathcal{D}$, because $\mathcal{D}^0 + \mathcal{D}^4 = \mathcal{P}^0 + \mathcal{P}^3$. The space

$$\mathcal{D}^1 + \mathcal{D}^3 = \mathcal{D}^1 + \mathcal{D}^1\mathbf{i}$$

is closed under multiplication by $\mathbf{i}$, and hence by complex scalars, and will be called the space of *complex vectors*. A typical complex vector has the form $a + b\mathbf{i}$, $a, b \in \mathcal{D}^1$. The space $\mathcal{D}^2$ of bivectors is closed under multiplication by $\mathbf{i}$ and hence it is a $\mathbf{C}$-space as well. A typical bivector has the form $\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i}$, with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$. Thus a *complex multivector* has the form

$$(\alpha + \beta\,\mathbf{i}) + (a + b\,\mathbf{i}) + (\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i})$$

$$\alpha, \beta \in \mathbb{R},\ \ a, b \in \mathcal{D}^1 = E_{1,3},\ \ \boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}.$$

Lorentz transformations. Given $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\mathbf{i}$, we have $\boldsymbol{z}^2 = \boldsymbol{x}^2 - \boldsymbol{y}^2 + 2(\boldsymbol{x}\cdot\boldsymbol{y})\mathbf{i} \in \mathbf{C}$ (we have used that $\mathbf{i}$ commutes with bivectors, that $\boldsymbol{xy} + \boldsymbol{yx} = 2(\boldsymbol{x}\cdot\boldsymbol{y})$, where the inner product is relative to $\mathcal{E}$, and that $\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{x}\cdot\boldsymbol{y} \in \mathbb{R}$). We will say that $\boldsymbol{z}$ is a *Lorentz bivector* if $\boldsymbol{z}^2 = \pm 1 = \epsilon$ and in this case we define the $\boldsymbol{z}$-rotor of amplitude $\alpha \in \mathbb{R}$ as

$$R = R_{\boldsymbol{z},\alpha} = e^{\alpha\boldsymbol{z}/2} = \cos_\epsilon(\alpha/2) + \boldsymbol{z}\sin_\epsilon(\alpha/2),$$

where $\cos_\epsilon$ and $\sin_\epsilon$ denote cosh and sinh if $\epsilon = 1$, cos and sin if $\epsilon = -1$. Note that $R_{\boldsymbol{z},\alpha}\,R_{-\boldsymbol{z},\alpha} = 1$ and hence $e^{-\alpha\boldsymbol{z}/2} = R^{-1}_{\boldsymbol{z},\alpha}$. Since $\tilde{\boldsymbol{z}} = -\boldsymbol{z}$, this also shows that $R^{-1} = \tilde{R}$.

Let $L = L_{\boldsymbol{z},\alpha}$ be the automorphism of $\mathcal{D}$ defined by

$$L(x) = RxR^{-1} = Rx\tilde{R}.$$

The map L has the property that $L\mathcal{D}^1 = \mathcal{D}^1$, for if $a \in \mathcal{D}^1$, then

$$\widehat{La} = \hat{R}\hat{a}\hat{\tilde{R}} = -Ra\tilde{R} \ \text{ and } \ \widetilde{La} = \tilde{\tilde{R}}\tilde{a}\tilde{R} = Ra\tilde{R} = La,$$

and so we can apply the observation at the end of the Involutions paragraph.

Furthermore, L is a proper Lorentz isometry (in symbols, $L \in O^+_\eta$), for

$$\eta(La) = (La)^2 = RaR^{-1}\,RaR^{-1} = R\,a^2R^{-1} = a^2 = \eta(a), \text{ and}$$

$$\det(L)\mathbf{i} = L(\mathbf{i}) = R\mathbf{i}R^{-1} = \mathbf{i}, \text{ which implies } \det(L) = 1.$$

For practical computations, note that

$$L(a) = RaR^{-1} = (\cos_\epsilon(\alpha/2) + \boldsymbol{z}\sin_\epsilon(\alpha/2))\,a\,(\cos_\epsilon(\alpha/2) - \boldsymbol{z}\sin_\epsilon(\alpha/2)).$$

Example. Let $\boldsymbol{u}$ be a unit vector of $\mathcal{E}$, so that $\boldsymbol{u}^2 = 1$, and write $u = \boldsymbol{u}\gamma_0 \in \langle\gamma_1, \gamma_2, \gamma_3\rangle$.

Note that $\boldsymbol{u}$ is the relative vector of u, for $u \wedge \gamma_0 = u\gamma_0 = \boldsymbol{u}$. Since $\boldsymbol{u}$ is a Lorentz vector, we can consider $L = L_{\boldsymbol{u},\alpha}$. We will see that L is the Lorentz boost in the direction u of rapidity α (these terms are explained in the reasoning that follows). First, let us find $L\gamma_0$. Since γ_0 anticommutes with $\boldsymbol{u}$,

$$L(\gamma_0) = R^2_{\boldsymbol{u},\alpha}(\gamma_0) = e^{\alpha\boldsymbol{u}}\gamma_0 = \cosh(\alpha)\gamma_0 + \sinh(\alpha)u.$$

Now we have

$$\begin{aligned} Lu &= e^{\alpha\boldsymbol{u}/2}ue^{-\alpha\boldsymbol{u}/2} = e^{\alpha\boldsymbol{u}/2}\boldsymbol{u}e^{\alpha\boldsymbol{u}/2}\,\gamma_0 = e^{\alpha\boldsymbol{u}}u \\ &= (\cosh(\alpha) + \sinh(\alpha)\,\boldsymbol{u})u = \sinh(\alpha)\gamma_0 + \cosh(\alpha)u. \end{aligned}$$

Since vectors $a \in \langle\gamma_0, u\rangle^{\perp}$ commute with $\boldsymbol{u}$, these vectors satisfy $La = a$.
Finally letting $\alpha = \tanh(\beta)$, then

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$$

where $\gamma = \cosh\alpha = (1-\beta^2)^{-1/2}$, which agrees with the so called *Lorentz boost* in the direction and *relativistic speed* β. In this context, α is usually called the *rapidity parameter*.

Example. Let $\boldsymbol{u}$ be a unit vector of $\mathcal{E}$, and consider $\boldsymbol{z} = \boldsymbol{u}\mathbf{i}$. Since $\boldsymbol{z}^2 = -\boldsymbol{u}^2 = -1$, $\boldsymbol{z}$ is a Lorentz bivector with $\epsilon = -1$. If we let $L = L_{\boldsymbol{z},\alpha}$, then γ_0 commutes with $\boldsymbol{z}$ and hence

$$L(\gamma_0) = e^{\alpha\boldsymbol{z}/2}\,\gamma_0\,e^{-\alpha\boldsymbol{z}/2} = e^{\alpha\boldsymbol{z}/2}e^{-\alpha\boldsymbol{z}/2}\,\gamma_0 = \gamma_0.$$

It follows that L is a rotation of the space $\langle\gamma_1, \gamma_2, \gamma_3\rangle$. Since u commutes with $\boldsymbol{z}$, u is fixed by L and hence $\langle u\rangle$ is the axis of the rotation. And if $v \in \langle\gamma_1, \gamma_2, \gamma_3\rangle$ is orthogonal to u, then v anticommutes with $\boldsymbol{z}$ and

$$L(v) = e^{\alpha\boldsymbol{z}}\,v = e^{\alpha\boldsymbol{u}\mathbf{i}}\,v = \cos(\alpha)v + \boldsymbol{u}\mathbf{i}v\,\sin(\alpha).$$

This shows that the amplitude of the rotation L is α, because $\boldsymbol{u}\mathbf{i}v$ is orthogonal to u and to v (it is a plain computation to see that $\boldsymbol{u}\mathbf{i}v$ anticommutes with u and with v).

The gradient operator. For applications of $\mathcal{D}$, particularly to physics, we also need the *gradient operator* $\partial = \partial_{\mathcal{D}}$.[10] It is defined as $\gamma^{\mu}\partial_{\mu}$.[11] Owing to the Schwarz rule, the ∂_{μ} *behave as scalars* and so ∂ *behaves as a vector*. Thus we have,

10 There is not a generally accepted notation for this operator. Among the symbols used, there are ∇ and $\square$.

11 We follow Einstein's convention of assuming a summation over a repeated index, and that Greek indices vary from 0 to 3. In contrast, indices that vary from 1 to 3 are denoted by Latin characters.

for any multivector x,

$$\partial x = \partial \cdot x + \partial \wedge x.$$

The relative version is

$$\boldsymbol{\partial} = \partial \wedge \gamma_0 = \gamma^k \wedge \gamma_0\, \partial_k = -\sigma_k\, \partial_k = -\nabla,$$

where $\nabla = \sigma_k \partial_k$ is the gradient operator in the (Euclidean) relative space. Then we have:

$$\partial\gamma_0 = \partial_0 + \boldsymbol{\partial} = \partial_0 - \nabla \text{ and } \gamma_0\partial = \partial_0 - \boldsymbol{\partial} = \partial_0 + \nabla.$$

The Maxwell-Riesz equation. In $\mathcal{D}$, the electromagnetic field is represented by a bivector F and the charge-current density by a vector j. The *Maxwell-Riesz equation* (MR) for F is

$$\partial F = j.$$

Our next task is to indicate how this equation is equivalent to the textbook Maxwell's equations. For the advantages of this formulation, see Doran-Lasenby 2003.

Using the split $\mathcal{D}^2 = \mathcal{E} + \mathbf{i}\mathcal{E}$, we can write F in relative terms:

$$F = \boldsymbol{E} + \mathbf{i}\boldsymbol{B},\ \boldsymbol{E}, \boldsymbol{B} \in \mathcal{E}.$$

Thus F appears to be constituted, in the frame $\boldsymbol{\gamma}$, of the *electric* and *magnetic* (relative) vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$. We also have, writing $\rho = j \cdot \gamma_0$ (*charge density*) and $\boldsymbol{j} = j \wedge \gamma_0$ (relative *current density*), that

$$j\gamma_0 = j \cdot \gamma_0 + j \wedge \gamma_0 = \rho + \boldsymbol{j}.$$

From this we get that

$$j = (\rho + \boldsymbol{j})\gamma_0 \text{ and } \gamma_0 j = \rho - \boldsymbol{j}.$$

Now we can proceed to show that the MR equation is equivalent to the Maxwell's equations for $\boldsymbol{E}$, $\boldsymbol{B}$, ρ and $\boldsymbol{j}$. Multiply the equation MR by γ_0 on the left to obtain:

$$(\partial_0 + \nabla)(\boldsymbol{E} + \mathbf{i}\boldsymbol{B}) = \rho - \boldsymbol{j}.$$

This expands to

$$\partial_0\boldsymbol{E} + \nabla\boldsymbol{E} + \mathbf{i}(\partial_0\boldsymbol{B} + \nabla\boldsymbol{B}) = \rho - \boldsymbol{j}.$$

But we also have $\nabla\boldsymbol{E} = \nabla \cdot \boldsymbol{E} + \nabla \wedge \boldsymbol{E}$ and $\nabla\boldsymbol{B} = \nabla \cdot \boldsymbol{B} + \nabla \wedge \boldsymbol{B}$, and so the last equation gives

$$\partial_0\boldsymbol{E} + \nabla \cdot \boldsymbol{E} + \nabla \wedge \boldsymbol{E} + \mathbf{i}(\partial_0\boldsymbol{B} + \nabla \cdot \boldsymbol{B} + \nabla \wedge \boldsymbol{B}) = \rho - \boldsymbol{j}.$$

This equation *is equivalent to the four equations obtained by equating the components for the grades* 0 *to* 3, which are:

$$\nabla \cdot \boldsymbol{E} = \rho$$

$$\partial_0 \boldsymbol{E} + \mathbf{i}(\nabla \wedge \boldsymbol{B}) = -\boldsymbol{j}$$

$$\nabla \wedge \boldsymbol{E} + \mathbf{i}\partial_0 \boldsymbol{B} = 0$$

$$\mathbf{i}(\nabla \cdot \boldsymbol{B}) = 0$$

Using that $\mathbf{i}(\nabla \wedge \boldsymbol{B}) = -\nabla \times \boldsymbol{B}$ and $\mathbf{i}(\nabla \wedge \boldsymbol{E}) = -\nabla \times \boldsymbol{E}$, the four equations can be written as:

$\nabla \cdot \boldsymbol{E} = \rho$	(Gauss law for $\boldsymbol{E}$)
$\nabla \times \boldsymbol{B} - \partial_0 \boldsymbol{E} = \boldsymbol{j}$	(Ampère-Maxwell equation)
$\partial_0 \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0$	(Faraday induction law)
$\nabla \cdot \boldsymbol{B} = 0$	(Gauss law for $\boldsymbol{B}$)

Electromagnetic potential. An *electromagnetic potential* is a vector field A such that

$$\partial A = F.$$

Let $\phi = A \cdot \gamma_0$ (*scalar potential*) and $\boldsymbol{A} = A \wedge \gamma_0$ (*vector potential*). Then $A\gamma_0 = \phi + \boldsymbol{A}$ and $\gamma_0 A = \gamma_0 (\phi + \boldsymbol{A})\gamma_0 = \phi - \boldsymbol{A}$. So we have:

$$\boldsymbol{E} + \mathbf{i}\boldsymbol{B} = F = \partial A = \partial\gamma_0\gamma_0 A = (\partial_0 - \nabla)(\phi - \boldsymbol{A}).$$

Equating components of the same grade, we find that this expression is equivalent to the following three equations (the pseudoscalar components vanish):

$\partial_0 \phi + \nabla \cdot A = 0$	(Lorentz gauge equation)
$\boldsymbol{E} = -(\nabla\phi + \partial_0 \boldsymbol{A})$	(expression of $\boldsymbol{E}$ in terms of the potentials)
$\boldsymbol{B} = \nabla \times \boldsymbol{A}$	(expression of $\boldsymbol{B}$ in terms of the vector potential)

These formulas agree with the textbook relations giving the electric and magnetic fields in terms of the scalar and vector potentials (in the Lorentz gauge).

Dirac equation. In the original setting, *Dirac spinors* are elements of $\mathbb{C}^4$. The careful study of the Dirac equation with a geometric algebra perspective (we refer to the excellent paper Hestenes 2003 for details) leads to the conclusion that a Dirac spinor is best represented as an element $\psi \in \mathcal{D}^+$. Moreover, in this approach the Dirac equation for the electron in an electromagnetic field of potential A takes the form

$$\partial\psi = (m\psi\gamma_0 + eA\psi)\gamma_2\gamma_1,$$

where m and e are the mass and charge of the electron, respectively. It is written purely in terms of GA and its study leads to its deep geometrical and physical significance.

7.- Conclusion.

Reading again the long quotation of Leibniz's correspondence with Huygens included in the second section of this paper, we realize that Leibniz vision is realized in GA not only for the geometry of ordinary space (the algebra $\mathcal{G}_3$), but also for other spaces of fundamental relevance for mathematics and physics, as illustrated here with the study of the Dirac algebra $\mathcal{D}$.

"I believe that by this method one could treat mechanics almost like geometry", says Leibniz, and this is found to be amply achieved in Hestenes 1990 (we can even drop the 'almost'). The treatment of rotations in ordinary space using $\mathcal{G}_3$, particularly the determination of the composition of two rotations, confirm the foresight that "this new characteristic cannot fail to give the solution, the construction, and the geometric demonstration all at the same time, and in a natural way in one analysis, that is, through determined procedure". It is important to remark that GA comes with the big bonus that an analogous procedure works in many other situations in geometry, physics and engineering (illustrated here with the description of Lorentz transformations, but widely confirmed by the available literature). A final remark is that the language of mathematical structures initiated by Grassmann plays the role of Leibniz's *characteristica universalis*. It is in that realm that GA is grounded as a most remarkable jewel.

8.- Notes.

N1. Notice that if $v_1 = \alpha_{11}u_1 + \alpha_{12}u_2$ and $v_2 = \alpha_{21}u_1 + \alpha_{22}u_2$ is another orthonormal basis, then $\langle v_1, v_2\rangle = \langle u_1, u_2\rangle = \mathcal{G}_2^1$, and since

$$v_1v_2 = \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\, u_1u_2 = \pm\, u_1u_2$$

(for $\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = q(v_1, v_2) = 0$ and $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \det_{u_1, u_2}(v_1, v_2) = \pm 1$), we also have $\langle v_1\, v_2\rangle = \langle u_1u_2\rangle = \mathcal{G}_2^2$.

N2. Here is a brief summary of the Grassmann exterior algebra. Let E is a real vector space of finite dimension n. The ingredients of the j-th *exterior power* of E (j a non-negative integer) are a vector space $\Lambda^j E$ and a multilinear alternating map $\wedge_j : E^j \to \Lambda^j E$. The defining property of these ingredients is that for any multilinear alternating map $f: E^j \to F$, where F is any vector space, there exists a unique *linear* map $\bar{f} : \Lambda^j E \to F$ such that

$$\bar{f}\left(\wedge_j (a_1, \cdots, a_k)\right) = f(a_1, \cdots, a_j).$$

From this it is straightforward to see that there is a unique bilinear map

$$\wedge_{j,k} : \Lambda^j E \times \Lambda^k E \to \Lambda^{j+k} E$$

such that

$$\wedge_{j,k}\left(\wedge_j (a_1, \cdots, a_j), \wedge_k (a_{j+1}, \cdots, a_{j+k})\right) = \wedge_{j+k} (a_1, \cdots, a_j, a_{j+1}, \cdots, a_{j+k})$$

Let $\Lambda E = \Lambda^0 E \oplus \Lambda^1 E \oplus \cdots \oplus \Lambda^n E$ (note that $\Lambda^k E = 0$ if $k > n$). Given $x, y \in \Lambda E$, its *exterior product* $x \wedge y$ is defined as $\sum_{j,k} \wedge_{j,k}(x_j, y_k)$. It is bilinear, unital, and associative, and the (Grassmann) *exterior algebra* of E is ΛE endowed with the exterior product $\wedge$.

The elements of ΛE are called *multivectors*. The multivectors of $\Lambda^k E$ are called *k-vectors*. For $k = 0$, $\Lambda^0 E = \mathbb{R}$, and its elements are called *scalars*. For $k = 1$, $\Lambda^1 E = E$, and its elements are called *vectors*. Instead of 2-vectors or 3-vectors we usually say *bivectors* and *trivectors*, respectively. If $x_k \in \Lambda^k E$ and $x_l \in \Lambda^l E$, then

$$x_k \wedge x_l \in \Lambda^{k+l} E \text{ and } x_k \wedge x_l = (-1)^{kl} x_l \wedge x_k.$$

If $a_1, \cdots, a_k$ are vectors, then $a_1 \wedge \cdots \wedge a_k = \wedge_j (a_1, \cdots, a_k) \in \Lambda^k E$ and a key property of the exterior algebra is that $a_1 \wedge \cdots \wedge a_k = 0$ *if and only* $a_1, \ldots, a_k$ *are linearly dependent*. It follows that if $a_1 \wedge \cdots \wedge a_k \neq 0$ (in which case we say that $a_1 \wedge \cdots \wedge a_k$ is a *k-blade*), then it determines the linear span $\langle a_1, \ldots, a_k \rangle \subseteq E$ as the vector subspace of E formed by the vectors a such that $a \wedge a_1 \wedge \cdots \wedge a_k = 0$. This leads to the fundamental representation of linear subspaces of dimension k of E as k-blades, up to a non-zero scalar multiple.

Let $e_1, \ldots, e_n$ be a basis of E. Given a *multiindex* $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, let

$$e_{\hat{I}} = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda^k E.$$

These blades form a basis of $\Lambda^k E$ and consequently

$$\dim \Lambda^k E = \tbinom{n}{k} \text{ and } \dim \Lambda E = 2^n.$$

N3. Strictly speaking, so far we have only shown that if $\mathcal{G}_2$ exists, then it is unique up to a canonical isomorphism, and that it has dimension 4. The existence is a consequence of the Pauli representation, but it can be proven in general without resorting to matrices. This is done in many references, and in particular in Xambó-Descamps 2018. For $\mathcal{G}_3$, see next note.

N4. The linear independence of 1, u_1, u_2, u_3, u_1u_2, u_1u_3, u_2u_3, $u_1u_2u_3$ can be seen by the method used for E_2 (Riesz's method). Suppose we have a linear relation

$$\lambda + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \mu_1 u_2 u_3 + \mu_2 u_1 u_3 + \mu_3 u_1 u_2 + \mu u_1 u_2 u_3 = 0.$$

Multiplying by u_1 from the left and from the right, we get

$$\lambda + \lambda_1 u_1 - \lambda_2 u_2 - \lambda_3 u_3 - \mu_1 u_2 u_3 - \mu_2 u_1 u_3 - \mu_3 u_1 u_2 + \mu u_1 u\ u_3 = 0.$$

Adding the two, we obtain

$$\lambda + \lambda_1 u_1 + \mu_1 u_2 u_3 + \mu u_1 u_2 u_3 = 0.$$

Now multiply by u_2 from the left and from the right, to get

$$\lambda - \lambda_1 u_1 - \mu_1 u_2 u_3 + \mu u_1 u_2 u_3 = 0,$$

and hence $\lambda + \mu u_1 u_2 u_3 = 0$. Since $(u_1u_2u_3)^2 = -1$, and λ, μ are real, we must have $\mu = \lambda = 0$. Thus $\lambda_1 u_1 + u_1 u_2 u_3 = 0$. Multiplying by u_1 from the left, we conclude, as in the previous step, that $\lambda_1 = \mu_1 = 0$. So we are left with the relation

$$\lambda_2 u_2 + \lambda_3 u_3 + \mu_2 u_1 u_3 + \mu_3 u_1 u_2 = 0.$$

Repeat the game: multiplying by μ_2 from the left and the right, we easily conclude that $\lambda_2 = \mu_2 = 0$ and then $\lambda_3 = \mu_3 = 0$ follows readily.

N5. Indeed, $\boldsymbol{i}_k^2 = u_k \boldsymbol{i} u_k \boldsymbol{i} = u_k^2 \boldsymbol{i}^2 = -1$ ($k = 1,2,3$) and if (j,k,l) is a cyclic permutation of (1,2,3), then $\boldsymbol{i}_j \boldsymbol{i}_k = u_j \boldsymbol{i} u_k \boldsymbol{i} = \boldsymbol{i} u_j u_k \boldsymbol{i} = u_l \boldsymbol{i} = \boldsymbol{i}_l$.

N6. With the same notations as in **N2**, the extension of a metric q to ΛE is determined by requiring that $q(x, y) = 0$ is $x \in \Lambda^j E$, $y \in \Lambda^k E$ and $j \neq k$ and that $q(A, B) = G(A, B)$ if $A, B \in \Lambda^k E$ are blades, where $G(A, B)$ denotes the *Gram determinant* of A and B:

$$G(a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k) = \det \left(q(a_i, b_j)\right)_{1 \leq i, j \leq k}$$

In particular, $q(A) = G(A)$, where $G(A) = G(A, A)$. For example,

$$q(a_1 \wedge a_2) = q(a_1)q(a_2) - q(a_1, a_2)^2.$$

N7. Let q be a metric of a real space E of dimension n. Let $u_1, \ldots, u_n$ be an q-orthonormal basis of E. Let $\mathcal{G}$ be the geometric algebra of (E, q). For any non-negative integer r and any sequence $J = j_1, \ldots, j_r \in \{1, \ldots, n\}$, write

$$u_J = u_{j_1,\dots,j_r} = u_{j_1} \cdots u_{j_r}.$$

Then the u_J with $j_1 < \cdots < j_r$, $0 \le r \le n$, form a basis of $\mathcal{G}$ (of $\mathcal{G}^r$ for a fixed r), and Artin's formula holds for this basis:

$$u_J u_K = (-1)^{\iota(J,K)}\, q(u_{J \cap K})\, u_{J \Delta K}.$$

The proof is straightforward. We can rearrange the product $u_J u_K$ until the sequence of indices is in non-decreasing order. This produces $\iota(J,K)$ sign changes, hence the sign $(-1)^{\iota(J,K)}$; for each $l \in J \cap K$, we get a factor $u_l^2 = q(u_l)$, hence altogether a factor $q(u_{J \cap K})$, and the remaining indices form $J \Delta K$ in increasing order, to which corresponds $u_{J \Delta K}$.

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